7. MAZ'YA V.G. and PLAMENEVSKII B.A., On the coefficients in the asymptotic form of the solutions of elliptical boundary-value problems in regions with conical points. Math. Nachr., 76, 1977.
8. CHEREPANOV G.P., Mechanics of Brittle Fracture. Nauka, Moscow, 1974.
9. GOL'DENVEIZER A.L., Constructing an approximate theory of flexure of a plate using the method of asymptotic integration of equations of the theory of elasticity. PMM 26, 4, 1962.
10. ZINO I.E. and TROPP E.A., Asymptotic Methods in Problems of the Theory of heat Conduction and Thermoelasticity. Iz-vo LGU, Leningrad, 1978.
11. BENTHEM J.P. and KOITER W.T., Asymptotic approximation to crack problems, Mechanics of fracture. V. 1. Methods of analysis and solutions of crack problems. Leyden: Noordhoff Intern. Publ., 1973.
12. MAZ'YA V.G., NAZAROV S.A. and PLAMENEVSKII B.A., Computing the asymptotic forms of the intensity factors when the angle or conical points approach each other. Zh. vychisl. Mat. i mat. Fiz., 23, 2, 1983.
13. ZLATIN A.N. and KHRAPKOV A.A., A semi-infinite crack parallel to the boundary of an elastic half-plane. Dokl. Akad. Nauk SSSR, 291, 4, 1986.
14. SLEPYAN L.I., Mechanics of Cracks. Sudostroyeniye, Leningrad, 1981.
15. MIKHAILOV A.M., Generalization of the beam approach to the problems of the theory of cracks. PMTF, 3, 1969.
16. CHEREPANOV G.P., Mechanics of Fracture of Composite Materials. Nauka, Moscow, 1983.
17. RICE J.R., A path independent integral and the approximate analysis of strain concentration by notches and crack, Trans. ASME. Ser. E.J. Appl. Mech, 35, 2, 1968.
18. SEDOV L.I., Mechanics of a Continuous Medium, 2, Nauka, Moscow, 1976.

Translated by L.K.

PMM U.S.S.R., Vol.54, No.1, pp. 115-133, 1990
Printed in Great Britain

0021-8928/90 $10.00+0.00

EXTREMAL CRITERIA OF THE STABILITY OF CERTAIN MOTIONS*

I.I. BLEKHMAN and O.Z. MALAKHOVA

The so-called extremal criteria of the stability of certain types of motion were formulated in a number of publications /1, 3, 4, 7-10, 13, 16, 23-25, 31-34, 36-40, 44/. However, until now, the connection between these criteria has not been discussed, nor the problem of the possibility of extending them to embrace the wider classes of systems and motions considered. In a number of cases it might be found that the results of various investigations are contradictory.

In this connection the present paper combines a comparative survey of the work dealing with extremal criteria of stability, with a derivation (in cases when it was not already done) of the criteria in question in a unique manner, using the Poincare-Lyapunov small-parameter method. It should be noted that the same results can be obtained, under somewhat different assumptions, by the method of direct separation of motions. Three classes of systems are specified for which the extremal criteria of stability have been successfully established up to the present time. The basic results are given in the form of theorems. The applications of extremal criteria to the problem of deriving a general justification for the tendency for certain classes of weakly connected dynamic objects to synchronize, to the problems of designing new

**Prikl. Matem. Mekhan.*, 54, 1, 142-161, 1990

vibrational devices and technologies, to generalizing the principle of selfbalancing of non-equilibrium rotors and the problem of resonances (synchronisms) in the motions of celestial objects are discussed.

1. On the extremal criteria of stability and of dynamic systems that are potential in the mean. General assumptions concerning the differential systems studied. All the criteria of stability discussed above are characterized by the fact that for the specific systems and their classes of motions, a sufficiently smooth function $D(\alpha_1, \ldots, \alpha_k)$ of a certain number of parameters $\alpha_1, \ldots, \alpha_k$ ($k \leqslant 2n$, where $2n$ is the order of the system) can be found such, that the stable motions of the given class /13/ correspond, or can correspond, to the minimum (and sometimes the maximum) points of this function. In other words, in the cases in question we have an analogue of the Lagrange-Dirichlet theorem on the stability of equilibrium positions, and this presents great advantages during the investigations. It is also essential that the function D itself, which we shall call the potential function, as well as the parameters $\alpha_1, \ldots, \alpha_k$, can be clearly interpreted in terms of the characteristics of the system and the motions in question. In all the criteria under discussion the function D or its "fundamental part" represents, in a clearly defined manner, the averaged Lagrangian, Hamiltonian and the force function of the system or its part, and the parameters $\alpha_1, \ldots, \alpha_k$ represent the approximate or exact values of the generalized coordinates (see below).

The conditions of existence of a potential function are sufficiently strict, and its presence is by no means always established. In this connection we shall consider in Sect.2-4 three relatively wide and important classes of systems for which the function D was found. The basic results are formulated in the form of mathematical assertions which, as a rule, has not been done by the authors. Also, in order to clarify the point of the problem we shall show, albeit purely schematically, two routes leading to the appearance of a potential function D corresponding to two methods of investigation, namely to the Poincaré-Lyapunov method and the method of averaging.

As we know, when the Poincaré method is applied to the case in which the generating system can have a family of periodic or almost periodic solutions depending on the parameters $\alpha_1, \ldots, \alpha_k$, then the corresponding solutions of the initial systems may correspond to the values of these parameters satisfying a certain system of equations

$$P_s(\alpha_1, \ldots, \alpha_k) = 0 \quad (s = 1, \ldots, k) \tag{1.1}$$

where the functions P_s are expressed in terms of the right-hand sides of the initial equations and of the generating solution (see Sect.2-4). Further, we shall show that when specific assumptions are made concerning the form of the solutions of the equations in variations corresponding to the generating system and generating solution /5, 6, 27, 34/, that a specified solution of Eq.(1.1) has indeed a corresponding asymptotically stable solution of the initial system, provided that all roots $\varkappa$ of the algebraic k-the degree equation (δ_{sj} is the Kronecker delta)

$$|\partial P_i/\partial\alpha_j - \delta_{ij}\varkappa| = 0 \quad (i, j = 1, \ldots, k) \tag{1.2}$$

have negative real parts. When we have at least one root with a positive real part, then the corresponding solution is unstable, and the case of zero or purely imaginary roots needs additional investigation.

Let us now assume that a function $D(\alpha_1, \ldots, \alpha_k)$ exists, continuous with its first- and second-order derivatives and such that the following relations hold:

$$\partial D/\partial\alpha_i = -P_i(\alpha_1, \ldots, \alpha_k) \quad (i = 1, \ldots, k) \tag{1.3}$$

It then follows from what was said above that D is indeed a potential function.

Another route by which a potential function may appear characterizes the averaging methods /15, 30/. Thus, when the method of direct separation of motions /11/, resembling asymptotic methods, is used, we find that under certain conditions the approximate equations for the "slow" variables $\alpha_1, \ldots, \alpha_k$ can be written, in spite of the, generally speaking, non-conservative nature of the initial system, in the form

$$\mathrm{E}_{\alpha_i}(T) = -\partial D/\partial\alpha_i \quad (i = 1, \ldots, k) \tag{1.4}$$

$$\mathrm{E}_{\alpha_i} = \frac{d}{dt}\frac{\partial}{\partial\alpha_i^{\cdot}} - \frac{\partial}{\partial\alpha_i}$$

$$T = \frac{1}{2}\sum_{j=1}^{k}\sum_{m=1}^{k} a_{jm}(\alpha_1, \ldots, \alpha_k)\,\alpha_j^{\cdot}\alpha_m^{\cdot}$$

where T is the kinetic energy corresponding to slow motions and E is the Euler operator. We shall call the dynamic system that allows the formulation of equations of slow motions in the form (1.4), the system potential in the mean. Such a description can be justified by the fact noted above, that the potential function D is usually obtained as a result of an averaging operation.

The Thomson-Tate-Chetayev theorems imply that the role of the function D is also maintained when terms corresponding to dissipative forces /28/ occur in Eqs.(1.4).

In the case when the appropriate equations for the slow variables $a_1, \ldots, a_l;\ b_1, \ldots, b_l\ (l \leqslant n)$ are written in canonical form

$$a_i^{\cdot} = -\partial H/\partial b_i,\ b_i^{\cdot} = \partial H/\partial a_i\ (i = 1, \ldots, l) \tag{1.5}$$

the Hamiltonian function $H = H(a_1, \ldots, a_l;\ b_1, \ldots, b_l)$ plays the role of the potential function D. However, in this case the stable motions can correspond to strict minima as well as strict maxima of the function H.

It would appear that the first result concerning the extremal criteria of stability follows from the classical work of Poincaré /35/ who, however, dealt only with conservative systems. We shall pause briefly to consider his result. Poincaré dealt with equations of the form

$$x_i^{\cdot} = \partial H/\partial y_i,\ y_i^{\cdot} = -\partial H/\partial x_i\ (i = 1, \ldots, n) \tag{1.6}$$
$$H = H_0(x_1, \ldots, x_n) + \mu H_1(x_1, \ldots, x_n, y_1, \ldots, y_n) + \mu^2 H_2(x_1, \ldots, x_n, y_1, \ldots, y_n) + \ldots$$

where H is a 2π-periodic function of the variables $x_1, \ldots, x_n; y_1, \ldots, y_n;\ \mu$ is a small parameter.

When $\mu = 0$, Eqs.(1.6) have the solution

$$x_i = a_i,\ y_i = \omega_i t + \alpha_i \tag{1.7}$$

where a_i and α_i are integration constants and ω_i are functions of $a_1, \ldots, a_n$.

Let us now assume that at certain values of a_i the frequencies ω_i are multiple $\omega = 2\pi/T$, i.e. the solution (1.7) is synchronous with frequency ω and $\omega_1 = -\partial H_0/\partial x_1 \neq 0$. We shall also assume that for appropriate values of a_i the Hessian $|\partial^2 H_0/\partial x^2|$ is different from zero. Since the problem is selfsimilar, we can write $\alpha_1 = 0$. Then, if for certain values of $\alpha_2 = \alpha_2^*, \ldots, \alpha_n = \alpha_n^*\ \partial \bar{H}_1/\partial \alpha_i = 0\ (i = 2, \ldots, n)$ and $|\partial^2 \bar{H}_1/\partial \alpha^2| \neq 0$

$$(\bar{H}_1(a_i, \alpha_i) = \langle [H_1] \rangle)$$

then for sufficiently small $\mu \neq 0$ the initial system (1.6) will have a T-periodic solution which transforms, when $\mu = 0$, to the generating solution (1.7) with parameters $\alpha_1 = 0, \alpha_2 = \alpha_2^*, \ldots, \alpha_n = \alpha_n^*$. Here and henceforth the square brackets indicate that the expression within them is calculated from the generating solution, and

$$\langle \ldots \rangle = \frac{1}{T}\int_0^T \ldots dt$$

in the case of periodic functions, and

$$\langle \ldots \rangle = \lim_{T\to\infty} \frac{1}{T}\int_0^T \ldots dt$$

in the case of almost periodic functions.

In the course of investigating the stability of the periodic solution in question, Poincaré showed that the characteristic indices of this solution can be written in the form of an expansion in powers of $\sqrt{\mu}$: $\lambda = \lambda_1\sqrt{\mu} + \lambda_2\mu + \ldots$, with two characteristic indices always equal to zero.

In particular, when $n = 2$, then the following expression holds for the characeristic indices /35/:

$$\omega_1^2\lambda_1^2 = -\left.\frac{\partial^2 \bar{H}_1}{\partial \alpha_2^2}\right|_{\alpha_2=\alpha_2^*}\left(\omega_1^2\frac{\partial^2 H_0}{\partial x_2^2} - 2\omega_1\omega_2\frac{\partial^2 H_0}{\partial x_1 \partial x_2} + \omega_2^2\frac{\partial^2 H_0}{\partial x_1^2}\right) \tag{1.8}$$

From Poincaré's arguments, based on considering expression (1.8), it follows that the motions that are stable in the first approximation will have corresponding minimum or maximum points of the function $\bar{H}_1$.

The above fragment of the classical work of Poincaré is separated from the extremal criteria of stability established subsequently by a fairly long period of time. We shall see from the survey given below, that it was towards the end of the fifties that papers started to appear in which extremal criteria of stability, applicable to more complex, but preferably non-conservative systems, were formulated. We note that for many such systems the extremal criteria are found to be "stronger". They express the necessary as well as the sufficient conditions of Lyapunov stability, i.e. they represent, in fact, the criterion of stability. In the case of Hamiltonian systems the corresponding criteria express, generally speaking, only the sufficient conditions of stability to a first approximation with respect to the small parameter occurring in the equations in variations. The motion in these cases may be Lyapunov stable only when certain additional conditions are satisfied. Such a situation occurs, in particular, in the extract from Poincaré's work discussed above. The extremal criteria do not lose their meaning even in these cases, since they determine the selection of the constants of the generating solution to which the stable motions can correspond.

As regards the nature and smoothness of the functions occurring in the differential equations in question, it is sufficient to assume, in order to ensure the validity of all the results given below, that these equations can be represented in the form

$$x_s^{\cdot} = X_s(x_1, \ldots, x_n, t) + \mu f_s(x_1, \ldots, x_n, t, \mu)$$

where the right-hand sides are defined for all real values of t, for values of μ lying in a certain segment $[0, \mu_0]$, and for the values of $x_1, \ldots, x_n$, lying within some closed region G of variables space. In this region the right-hand sides of the equations are continuous in t, and the functions f_s can have continuous partial first-order derivatives in $x_1, \ldots, x_n$. In some cases these demands can be relaxed, and the right-hand sides can be either T-periodic or almost periodic in t, or they may not depend explicitly on this variable. In the case of almost periodic equations we assume that for any fixed μ from the segment $[0, \mu_0]$ and any almost periodic $x_1, \ldots, x_n$, belonging to the region G, the functions $X_s(x_1, \ldots, x_n, t)$ and $f_s(x_1, \ldots, x_n, t, \mu)$ will also be almost periodic in t.

All variables and parameters are assumed to be dimensionless.

2. Systems with selfsynchronizing objects. Integral criteria of stability (extremal properties) of synchronous motions. 2.1. Systems with almost uniform rotations; selfsynchronization of oscillation generators. The integral criterion of stability was formulated for systems with selfsynchronizing mechanical oscillation generators /7/ and proved using the Poincaré-Lyapunov small-parameter method /8/. In /9/ the criterion is generalized to systems with almost uniform rotations, and the corresponding results can be formulated as follows.

Let us assume that the equations of motion of the system with generalized coordinates φ_s $(s = 1, \ldots, k)$ and u_r $(r = 1, \ldots \nu)$, characterized by the Lagrange function L and non-conservative generalized forces Q_{φ_s} and Q_{u_r}, can be written in the form

$$I_s \varphi_s^{\cdot\cdot} + k_s(\varphi_s^{\cdot} - \sigma_s n_s \omega) = \mu \Phi_s \quad (s = 1, \ldots, k) \tag{2.1}$$

$$E_{u_r}(L) = Q_{u_r} \quad (r = 1, \ldots, \nu) \tag{2.2}$$

where I_s, k_s and ω are positive constants, $\sigma_s = \pm 1$, n_s are positive integers and $\mu > 0$ is a small parameter.

The functions $L, Q_{\varphi_s}, Q_{u_r}$ and Φ_s can depend on the generalized coordinates and velocities of the system, as well as on the time t. The functions are 2π-periodic in φ_s, and $2\pi/\omega$-periodic in t. Furthermore, the functions Q_{φ_s}, Q_{u_r} and Φ_s can also depend on μ, and the functions Φ_s are found from the condition that Eqs.(2.1) are identically equal to the corresponding group of Lagrange equations of the second kind. We shall call such systems , systems with almost uniform rotations, the coordinates φ_s will be called rotational, and u_r will be called oscillatory coordinates.

The generating equations corresponding to Eqs.(2.1) have the following family of solutions:

$$\varphi_s^0 = \sigma_s(n_s \omega t + \alpha_s) \tag{2.3}$$

depending on k arbitrary parameters $\alpha_1, \ldots, \alpha_k$. Let the generating equations corresponding to Eqs.(2.2) and solution (2.3) have, for any α_s, an asymptotically stable $2\pi/\omega$-periodic solution u_r°. Further, let a function $B = B(\alpha_1, \ldots, \alpha_k)$ exist, called the potential of averaged generalized forces and such that

$$\frac{\partial B}{\partial \alpha_s} = \sigma_s \langle [Q_{\varphi_s}] \rangle + \sum_{r=1}^{\nu} \left\langle \left[Q_{u_r} \frac{\partial u_r^{\circ}}{\partial \alpha_s} \right] \right\rangle$$

where, as before, the angle brackets indicate averaging over the period $2\pi/\omega$ and the square brackets mean that the expression contained within them is also calculated, for $\mu = 0$, for the generating solution.

Let us denote by $\Lambda = \Lambda(\alpha_1, \ldots, \alpha_k) = \langle [L] \rangle$ the mean value of the Lagrange function of the system calculated for the generating solution.

Under the conditions formulated above the following theorem holds: to every point of the coarse minimum of the function

$$D = D(\alpha_1, \ldots, \alpha_k) = -(\Lambda + B) \tag{2.4}$$

there corresponds, for sufficiently small values of μ a unique, asymptotically stable solution of the initial system (2.1), (2.2) transforming, for $\mu = 0$, to the generating solution, i.e. to a solution of the type

$$\varphi_s = \sigma_s(n_s \omega t + \alpha_s) + \psi_s(t, \mu), \quad u_r = u_r^{\circ}(t) + v_r(t, \mu) \tag{2.5}$$

where ψ_s and $v_r - 2\pi/\omega$ are periodic functions of t, vanishing at $\mu = 0$ (we shall call such functions synchronous). The absence of a minimum detected by analysing the second-order terms of the expansion of the function D in powers of α_s near the stationary point, indicates the instability of the corresponding synchronous solution, and other cases require additional investigation.

Here and henceforth we will use the term "coarse" to describe the strict extremum of the function which can be detected by analysing the second-order terms in the expansion of this function near the stationary point.

We will make some additional remarks.

1°. If $L, Q_{\varphi_s}, Q_{u_r}$ and Φ_s are analytic functions of generalized coordinates and velocities, and the functions Q_{φ_s}, Q_{u_r} and Φ_s also depend analytically on the small parameter μ, the solution (2.5) of system (2.1), (2.2) will be analytic in μ.

2°. In the case of a selfsimilar system the function D will depend only on the differences $\alpha_1 - \alpha_k, \ldots, \alpha_{k-1} - \alpha_k$, the assertion made above will refer to the minima in these differences, and we shall speak of asymptotic orbital stability.

3°. When $\partial B/\partial\alpha_s \ll \partial\Lambda/\partial\alpha_s$, and especially when $B = \text{const}$, we can write $D = -\Lambda$, i.e. the role of a potential function will be played by the averaged Lagrangian of the system calculated for the generating solution and taken with the opposite sign.

4°. The above theorem clearly implies that the conditions for a coarse minimum of the function D represent, with the assumptions stated, not only the sufficient conditions of stability, but also the "coarsely necessary" conditions in the sense that the absence of a minimum determined by analysing the second-order terms in the expansion of the function D near the stationary point, indicate the instability of the synchronous solution in question.

5°. If the asymptotic stability of the generating solution u_r° cannot be established by analysing the equation in variations for system (2.2) at $\mu = 0$, then in order to obtain the sufficient conditions of stability under the conditions of coarse minimum it is necessary to incorporate certain additional relations obtained by analysing the higher-order approximations. The same situation occurs in the case when system (2.1) is quasiconservative. The conditions which follow from the demand of a coarse minimum of D are fundamental in this case also, since it is precisely these conditions that determine the choice of the constants $\alpha_1, \ldots, \alpha_k$, to which the stable solutions can correspond.

6°. Let the Lagrange's function of the system be represented in the form

$$L = L^* + L^{(I)} + L^{(II)}$$

$$L^* = \sum_{s=1}^{k} L_s(\varphi_s^{\cdot}, \varphi_s) + \sum_{r=1}^{\nu} f_r(\varphi_1^{\cdot}, \ldots, \varphi_k^{\cdot}, \varphi_1, \ldots, \varphi_k) u_r^{\cdot} + \sum_{s=1}^{k} F_s(\varphi_s)$$

$$L^{(I)} = \frac{1}{2}\sum_{r=1}^{\nu}\sum_{j=1}^{\nu} a_{rj} u_r^{\cdot} u_j^{\cdot} - \frac{1}{2}\sum_{r=1}^{\nu}\sum_{j=1}^{\nu} b_{rj} u_r u_j$$

$$L^{(II)} = \Psi(\varphi_1^{\cdot}, \ldots, \varphi_k^{\cdot}; \varphi_1, \ldots, \varphi_k)$$

Here a_{rj} and b_{rj} are constants and L_s, f_r, F_r and Ψ are functions of the variables listed above, and L_s, f_r and F_r are 2π-periodic functions in φ_s. Also let $[Q_{u_r}] \equiv 0$, i.e. there are no conservative generalized forces in coordinates u_r in the generating approximation. Therefore, the corresponding systems are quasilinear and quasiconservative with respect to the oscillatory coordinates. In this case the following relations hold /9, 10/:

$$\partial\Lambda_s/\partial\alpha_s \equiv 0, \quad \partial\Lambda/\partial\alpha_s = \partial(\Lambda^{(II)} - \Lambda^{(I)})/\partial\alpha_s$$

$$\Lambda_s = \langle[L_s]\rangle, \quad \Lambda^{(I)} = \langle[L^{(I)}]\rangle, \quad \Lambda^{(II)} = \langle[L^{(II)}]\rangle, \quad \Lambda = \langle[L]\rangle$$

and the potential function can be written in the form

$$D = \Lambda^{(I)} - \Lambda^{(II)} - B \tag{2.6}$$

Expressions $L_s, L^{(I)}$ and $L^{(II)}$ are called /32/, respectively, the characteristic Lagrangians of the synchronizing objects, and the Lagrangians of the systems of active and passive linkages between the objects. It is remarkable that the quantities $\Lambda^{(I)}$ and $\Lambda^{(II)}$ occur in the expressions for D with opposite signs. In the problem of synchronizing the oscillation generators (unbalanced rotors) L_s are the Legrangians of the rotors not connected with each other, $L^{(I)}$ is the Lagrangian of the elastic solid on a support or of a system of elastically coupled solids on which the rotors are mounted, and the term $L^{(II)}$ is governed by the presence of direct links between the rotors in the form of elastic and damper elements.

If $L^{(II)} = 0$, $B = \text{const}$ and $\partial\Lambda^{(I)}/\partial\alpha_s \gg \partial B/\partial\alpha_s$, then we can put

$$D = \Lambda^{(I)} = \langle[L^{(I)}]\rangle \tag{2.7}$$

i.e. the potential function in this case will be a Lagrangian, averaged over a period, of an elastic solid on a support or of a system of solids carrying the rotors, and the function $L^{(I)}$

will be determined in the generating approximation $\varphi_s^\bullet, u_r^\bullet$. This result considerably simplifies the investigation of systems with selfsynchronizing unbalanced rotors and has important applications (see Subsections 5.2 and 5.3).

The results given above can also be arrived at using the method of direct separation of motions /10, 11/. However, when this method is used, apart from the assumptions made above, we must assume that the parameter $\varepsilon = 1/\omega$ is fairly small. The integral criterion of stability is obtained in this case from the condition of stability of the stationary solutions $\alpha_1 = \alpha_1^*, \ldots, \alpha_k = \alpha_k^*$ of the system of equations

$$I_s\alpha_s^{\cdot\cdot} + k_s\alpha_s^{\cdot} = \partial(\Lambda + B)/\partial\alpha_s \quad (s = 1, \ldots, k) \tag{2.8}$$

where, unlike expressions (2.3), the quantities α_s are "slowly varying" functions of time.

A proof of the integral criterion of stability based on the use of a variational relation, was given in /25/.

In /9, 38, 45, 46/ expressions, are given which can be useful in solving specific problems, for the averaged Lagrangian of the quasilinear system of supporting bodies in terms of so-called harmonic influence coefficients. The modified geometrical formulation of the integral criterion of stability given in /23/ is useful in solving a number of problems dealing with selfsynchronization of the oscillation generators.

2.2. Systems of quasiconservative objects, canonical systems. The integral criterion of stability for a system of weakly coupled quasiconservative objects was obtained in /37/, and the expression for the potential function has the form /9, 10/

$$D = -(\Lambda + B)\sigma \tag{2.9}$$

$$\sigma = \operatorname{sign} e_s(\omega), \quad e_s(\omega_s) = \omega_s^{-1} dh_s/d\omega_s \tag{2.10}$$

(ω_s is the frequency of an isolated, purely conservative s-th object and $h_s(\omega_s)$ is its energy constant).

It is clear that the sign in expression (2.9) is determined by the sign of the quantities $e_s(\omega)$. Depending on this sign, Nagayev lists strongly anisochronous objects ($e_s > 0$, rotating rotors are an example), weakly anisochronous objects ($e_s < 0$, point masses rotating about a fixed centre under a force of attraction are an example) and isochronous objects ($e_s = \infty$, such as linear oscillators for which ω_s does not depend on h_s, so that $d\omega_s/dh_s = 0$). Formula (2.9) holds under the condition that the nature of the isochronicity is the same for all objects.

In the case of systems with quasilinear active linkages when the assumptions concerning the nature of the passive linkages are sufficiently general, expression (2.9) can be written in the form

$$D = (\Lambda^{(I)} - \Lambda^{(II)} - B)\sigma \tag{2.11}$$

Here $\Lambda^{(I)}$ and $\Lambda^{(II)}$ are, as in (2.6), the generalized Lagrangians of the systems of active and passive linkages calculated in the generating approximation.

In the problem of the synchronization of quasiconservative objects the conditions of stability expressed by the integral criterion are only coarsely necessary (see note 4° earlier). The sufficient conditions were obtained in /33, 34, 38/.

Asymptotic methods and the method of integral manifolds were used to generalize the results of /32/ to the case /21/ of so-called incomplete synchronism (the corresponding concept was formulated in the same paper).

Formulas (2.9) and (2.11) refer to the case of essentially anisochronous objects. In /32/ it was shown that the corresponding integral criterion of stability can also be formulated in the case of almost isochronous objects, which requires a special investigation. The integral criterion was also proved in /16/ for the case of almost isochronous objects, namely quasilinear oscillations. The authors there assume that the Lagrangian of the system has the form

$$L(q, q^{\cdot}, t, \mu) = \frac{1}{2}\sum_{j=1}^{n}(q_j^{\cdot 2} - \omega_j^2 q_j^2) + \mu l_1(q, q^{\cdot}, t, \mu) \tag{2.12}$$

where $\mu > 0$ is a small parameter, and l_1 is a $T_1 = 2\pi/\omega$ -periodic function of time t. We will consider a motion close to resonant:

$$\omega_j^2 - \nu_j^2 = O(\mu), \quad \nu_j = \omega p_j/N \quad (j = 1, \ldots, n)$$

where p_j $(j = 1, \ldots, n)$ and N are positive integers.

The equations of motion have the form

$$q_r^{\cdot\cdot} + \nu_r^2 q_r = -\mu\left(\frac{d}{dt}\frac{\partial l_2}{\partial q_r^{\cdot}} - \frac{\partial l_2}{\partial q_r}\right) \tag{2.13}$$

$$l_2(q, q^{\cdot}, t, \mu) \equiv l_1(q, q^{\cdot}, t, \mu) + \mu^{-1} \sum_{j=1}^{n} \frac{q_i^2}{2} (\nu_j^2 - \omega_j^2)$$

The generating solution of Eqs.(2.13) is

$$q_r^\circ = a_r \cos \nu_r t + b_r/\nu_r \sin \nu_r t \quad (r = 1, \ldots, n) \tag{2.14}$$

(a_r and b_r are the initial values of q_r° and $q_r^{\circ\cdot}$ respectively), $T = 2\pi N/\omega$ -periodic in t.

Introducing the mean value of the Lagrangian (2.12) along the generating periodic solution (2.14)

$$\Lambda(a, b) = \frac{1}{T} \int_0^T L(q^\circ(t), q^{\circ\cdot}(t), t, \mu)\, dt \tag{2.15}$$

the following assertion was made in /16/: if the function $\Lambda(a, b)$ has a minimum or a maximum at the point $a_1 = a_1^\circ, \ldots, a_n = a_n^\circ;\ b_1 = b_1^\circ, \ldots, b_n = b_n^\circ$, then the point in question determines the periodic solution, stable to a first approximation; other stationary points require a special investigation.

We must note that in accordance with the arguments used in /16/ the phrase "stable to a first approximation" means merely stability to a first approximation with respect to small parameter μ. In order to solve the problem of stability to a first approximation in the usual sense of this term we must, in the present case, investigate the roots of the characteristic equation with an accuracy of up to higher orders of μ.

The extremal criterion formulated here with the refinement quoted above can be obtained, under certain additional constraints, using Malkin's theorem /27/ of the existence and stability of almost periodic (and in particular periodic) solutions of system of equations of the form

$$x_s^{\cdot} = X_s(x_1, \ldots, x_l, t) + \mu F_s(x_1, \ldots, x_l, t, \mu) \quad (s = 1, \ldots, l) \tag{2.16}$$

where X_s and F_s are almost periodic functions of t, and for any μ from the segment $[0, \mu_0]$ and arbitrary, almost periodic functions $x_j(t)$ $(j = 1, \ldots, l)$ lying in some region G of the space of variables $x_1, \ldots, x_l$, the functions $X_s[x_1(t), \ldots, x_l(t), t]$ and $F_s[x_1(t), \ldots, x_l(t), t, \mu]$ are also almost periodic in t. The assumptions regarding the smoothness of the functions X_s and F_s were given earlier.

Malkin's theorem establishes a correspondence between the almost periodic solutions of system (2.16) and the almost periodic solutions of the generating system

$$x_s^{\circ\cdot} = X_s(x_1^\circ, \ldots, x_l^\circ, t) \quad (s = 1, \ldots, l) \tag{2.17}$$

For every almost periodic solution of system (2.17) lying in the region G and depending on l arbitrary parameters $\alpha_1, \ldots, \alpha_l$

$$x_s^\circ = x_s^\circ(t, \alpha_1, \ldots, \alpha_l) \tag{2.18}$$

for which

$$P_s(\alpha_1, \ldots, \alpha_k) \equiv \left\langle \sum_{j=1}^{n} F_j(x_1^\circ, \ldots, x_l^\circ, t, 0)\, z_{js}^*(t) \right\rangle = 0 \tag{2.19}$$

and the algebraic l-th degree equation

$$|\,\partial P_s/\partial \alpha_j - \delta_{sj}\lambda\,| = 0 \quad (s, j = 1, \ldots, l) \tag{2.20}$$

has no roots with zero real parts, there exists, for sufficiently small μ, an almost periodic solution of system (2.16) which transforms, for $\mu = 0$, into the generating solution (2.18). This almost periodic solution will be asymptotically stable if all roots of Eq.(2.20) have negative real parts; if the real part of at least one root is positive, then the corresponding solution will be unstable.

Here z_{js}^* are solutions of the system coupled to the system of equations in variations, constructed for the generating system (2.17) and generating solution (2.18), and

$$\sum_{s=1}^{l} z_{sm} z_{sj}^* = \delta_{mj} = \begin{cases} 1, & m = j \\ 0, & m \neq j \end{cases} \tag{2.21}$$

where $z_{sm} = \partial x_s^\circ/\partial \alpha_m$ $(s, m = 1, \ldots, l)$ are almost periodic solutions of equations in variations corresponding to system (2.17) and solution (2.18).

We note that the proof of the theorem formulated above is based on the Krylov and Bogolyubov's transformation for almost periodic systems in standard form, and also on Bogolyubov's theorem on the correspondence between the solutions of the initial system and the stationary solutions of the first approximation equations in an infinite time interval, and on the relation between the stabilities of the solutions in question /15/. In the case of periodic systems the proof of the Malkin's theorem is based on the methods of Poincaré and Lyapunov /27/. If at the same time Eq.(2.20) has no zero roots, conditions (2.19) will be necessary and sufficient for periodic solution of system (2.16) to exist, transforming, when $\mu = 0$, into the generating solution. The periodic solution will be asymptotically stable, provided that all roots of Eq.(2.20) have negative real parts. When Eq.(2.20) has purely imaginary roots, we can speak only of the stability of periodic solutions of system (2.16) in the first approximation.

Using the above theorem, we put $q_j = x_j,\ q_j^{\cdot} = y_j\ (j = 1, \ldots, n)$ * (*The proof given below as well as the proofs in Sect.3 and 4 are all due to O.Z. Malakhova.) and write system (2.13) in the form

$$x_r^{\cdot} = y_r,\ y_r^{\cdot} = -\nu_r^2 x_r + \mu F_r(x, y, y^{\cdot}, t, \mu)\ (r = 1, \ldots, n) \tag{2.22}$$

where the functions

$$F_r = -\frac{d}{dt}\frac{\partial l_2}{\partial y_r} + \frac{\partial l_2}{\partial x_r} \quad (r = 1, \ldots, n) \tag{2.23}$$

are $T = 2\pi/\omega$-periodic in t.

When $\mu = 0$, system (2.22) has the family of $T = 2\pi N/\omega$-periodic solutions depending on $2n$ arbitrary parameters $a_1, \ldots, a_n; b_1, \ldots, b_n$:

$$x_r^{\circ} = a_r \cos \nu_r t + b_r \nu_r^{-1} \sin \nu_r t \tag{2.24}$$

$$y_r^{\circ} = -a_r \nu_r \sin \nu_r t + b_r \cos \nu_r t \ (r = 1, \ldots, n)$$

Eqs.(2.19) for determining the parameters of the generating solution will be written as follows:

$$P_s(a, b) \equiv \frac{1}{T}\int_0^T \sum_{j=1}^n z_{j+n,s}^* F_j(x^{\circ}, y^{\circ}, y^{\circ\cdot}, t, 0)\, dt = 0 \quad (s = 1, \ldots, 2n) \tag{2.25}$$

$$z_{j,s}^* = \delta_{js}\cos\nu_s t \quad z_{j,s+n}^* = \delta_{js}\nu_s \sin \nu_s t \tag{2.26}$$

$$z_{j+n,s}^* = -\delta_{js}\nu_s^{-1}\sin\nu_s t \quad z_{j+n,s+n}^* = \delta_{js}\cos\nu_s t$$

$$(j = 1, \ldots, n;\ s = 1, \ldots, n)$$

Taking relations (2.23), (2.24) and (2.26) into account, we can obtain

$$P_j(a, b) = -\frac{1}{T}\int_0^T \frac{\partial}{\partial b_j} l_2(q^{\circ}, q^{\circ\cdot}, t, 0)\, dt$$

$$P_{j+n}(a, b) = \frac{1}{T}\int_0^T \frac{\partial}{\partial a_j} l_2(q^{\circ}, q^{\circ\cdot}, t, 0)\, dt \quad (j = 1, \ldots, n)$$

and from this it follows that the solution of system (2.25)

$$a_1 = a_1^{\circ}, \ldots, a_n = a_n^{\circ};\ b_1 = b_1^{\circ}, \ldots, b_n = b_n^{\circ}. \tag{2.27}$$

represents a stationary point of the function $\Lambda(a, b)$.

In order to study the problem of the existence and stability of a periodic solution of system (2.22) becoming, at $\mu = 0$, the generating solution (2.24) where $a_r = a_r^{\circ},\ b_r = b_r^{\circ}\ (r = 1, \ldots, n)$, we shall construct the algebraic Eq.(2.20) as follows:

$$\begin{vmatrix} -\dfrac{\partial^2\Lambda}{\partial b_r \partial a_s} - \delta_{rs}\lambda & -\dfrac{\partial^2\Lambda}{\partial b_r \partial b_s} \\ \dfrac{\partial^2\Lambda}{\partial a_r \partial a_s} & \dfrac{\partial^2\Lambda}{\partial a_r \partial b_s} - \delta_{rs}\lambda \end{vmatrix} = 0 \tag{2.28}$$

$$(r = 1, \ldots, n;\ s = 1, \ldots, n)$$

(the derivatives of the function $\Lambda(a, b)$ are calculated at the point (2.27)). For sufficiently small μ system (2.22) will have a periodic solution transforming, at $\mu = 0$, into the generating solution, provided that the algebraic equation mentioned above has no zero roots. Eq.(2.28) clearly represents the characteristic equation of the system in variations corresponding to the solution (2.27) of system

$$a_r^{\cdot} = -\partial\Lambda/\partial b_r,\ b_r^{\cdot} = \partial\Lambda/\partial a_r\ (r = 1, \ldots, n) \tag{2.29}$$

Since system (2.29) is Hamiltonian, it follows that the purely imaginary roots of Eq. (2.28) will correspond to the stationary points of the function $\Lambda(a, b)$, corresponding to positions that are stable to a first approximation. Therefore, the conditions for the presence of the coarse minimum or maximum of $\Lambda(a,b)$ at some point represents the sufficient conditions for the existence of a periodic solution of system (2.13) stable to a first approximation with respect to the small parameter μ, and transforming at $\mu=0$ into the generating solution (2.14).

We note that unlike the formulation /16/ given above, here we have the additional requirement of the coarse extremum of the function $\Lambda(a,b)$ brought in by the use of Malkin's theorem.

It was stated without proof in /16/ that the criterion of stability discussed here can be extended to the general case of canonical systems with a Hamiltonian function, almost periodic in t.

The principle of the extremal character of resonant (or, using the terminology adopted above, of synchronous) motions was proposed as a hypothesis for the problem of plane rotation of a celestial object about its centre of mass, moving along an elliptical orbit around a centre of attraction /1, 3, 4/.

Let $U(\theta,t)$ be the force function of the system where θ is the angle of deviation of the axis of inertia of the body from the radius vector of the orbit. We will introduce the time-averaged value of the force function:

$$\langle U\rangle = \bar{U}(\theta_0,\theta_0^{\cdot}) = \lim_{t\to\infty}\frac{1}{t}\int_0^t U[\theta(\theta_0,\theta_0^{\cdot},t),t]\,dt \tag{2.30}$$

According to the above hypothesis the limit (2.30) exists, and reaches a maximum value on the set of initial data $\theta_0,\theta_0^{\cdot}$ corresponding to stable resonant motions. The results of a numerical experiment /1/ were used as an argument in support of the hypothesis.

At a later date a theorem was proved which confirmed, in general terms, the idea that synchronous (resonant) motions have extremal properties /2/. A periodic system was considered, which is somewhat more general than a canonical system (a system in which the phase volume is preserved) and it was shown that the necessary and sufficient condition for stable periodic or synchronous motion to exist is that there exists a function $K(x_0)$ of initial values x_0 of the phase variables x, with a strict maximum or minimum in these initial values. Moreover, the function in question is connected with some function of the phase coordinates and time $\varkappa(x,t)$ by an integral relation of the form (2.30). The question of determining the functions $\varkappa$ and K has, however, remained open. We note that according to the above hypothesis U and $\langle U\rangle$ will be such functions.

We also note that the theorem in /2/ agrees with the results of /43/.

It was stated in /22/ that averaging (2.30) along the exact and, in general, non-degenerate solutions is different from averaging along degenerate solutions (i.e. depending on a certain number of parameters) which are discussed in the remaining cases considered above. The results of averaging along the degenerate solutions is affected by the choice of the averaged function, while in the case of non-degenerate periodic solutions stable in the linear approximation we find that for any continuously differentiable periodic function $\varkappa(x,t)$

$$\lim_{T\to\infty}\frac{\partial}{\partial x_0}\left(\frac{1}{T}\int_0^T \varkappa(x(t,x_0),t)\,dt\right)=0$$

and in particular, if $K(x_0)$ exists and is a continuously differentiable function, then $K_{x_0}'=0$, i.e. x_0 is a stationary point of the function $K(x_0)$. On the other hand, this assertion, which was proved in /22/, does not contradict the theorem given in /2/ in which only the existence of the function $\varkappa(x,t)$ is referred to and according to which x_0 is not only the stationary point of the function $\varkappa(x,t)$, but also represents the minimum or maximum point of this function, and the theorem gives the necessary, as well as the sufficient conditions for stable periodic solutions to exist.

2.3. Systems with quasicyclic coordinates. An integral criterion of stability was obtained in /39/ for systems with quasicyclic coordinates

$$p_r^{\cdot} = Q_r \quad (r=1,\ldots,m) \tag{2.31}$$

$$\mathrm{E}_{q_{m+r}}(L) = Q_{m+r} \quad (r=1,\ldots,n-m) \tag{2.32}$$

$$L = T(q_{m+1},\ldots,q_n;\, q_1^{\cdot},\ldots,q_n^{\cdot}) - \Pi(q_{m+1},\ldots,q_n)$$

where L is Lagrange's function, $\mu>0$ is a small parameter, $p_r=\partial T/\partial q_r^{\cdot}$ $(r=1,\ldots,m)$ are the quasicyclic momenta, and $Q_{m+r}(q_{m+1},\ldots,q_n;q_1^{\cdot},\ldots,q_n^{\cdot})$ are the generalized non-conservative forces corresponding to the position coordinates and the generalized. The non-conservative forces corresponding to the quasicyclic coordinates are assumed to be represented in the form

$$Q_r = U_r(t) + \mu f_r - \mu h_r q_r^{\cdot}$$
$$U_r(t + 2\pi/\omega) = U_r(t), \ \langle U_r(t)\rangle = 0 \quad (r = 1, \ldots, m)$$

where f_r, h_r are constants and $(h_r > 0)$.

The generating equations corresponding to Eqs.(2.31) have a family of $2\pi/\omega$-periodic solutions depending on m arbitrary constants $\alpha_1, \ldots, \alpha_m$:

$$p_r^\circ = \alpha_r + V_r(t) \quad (r = 1, \ldots, m)$$
$$(V_r^{\cdot} = V_r, \ \langle V_r\rangle = 0) \tag{2.33}$$

Let us assumed that the equations

$$[\mathrm{E}_{q_{m+r}}(L_R) - Q_{m+r}] = 0 \quad (r = 1, \ldots, n - m)$$

where

$$L_R = \left(T - \sum_{r=1}^{m} p_r q_r^{\cdot}\right|_{q_r^{\cdot} = q_r^{\cdot}(p_1, \ldots, p_m^{\cdot}; q_{m+1}, \ldots, q_n; q_{m+1}^{\cdot}, \ldots, q_n^{\cdot})} - \Pi$$

is the kinetic Routh potential have, at any $\alpha_1, \ldots, \alpha_m$, an asymptotically stable isolated $2\pi/\omega$-periodic solution

$$q_{r+m} = \overset{\circ}{q}_{r+m}(t, \alpha_1, \ldots, \alpha_m) \quad (r = 1, \ldots, n - m) \tag{2.34}$$

and, that the following relations hold:

$$\sum_{s=1}^{n-m} \left\langle \left[Q_{m+s} \frac{\partial q_{m+s}}{\partial \alpha_r} \right] \right\rangle = 0 \quad (r = 1, \ldots, m) \tag{2.35}$$

(here, as before, the functions in the square brackets are calculated using the generating solution (2.33)).

With the above assumptions we can make the following assertion; when μ are sufficiently small, we have, for every point of the coarse minimum of the potential function

$$D(\alpha_1, \ldots, \alpha_m) = -\langle [L_R]\rangle - \sum_{r=1}^{m} \frac{f_r}{h_r} \alpha_r \tag{2.36}$$

a corresponding, asymptotically stable periodic solution of system (2.31), (2.32) becoming, at $\mu = 0$, a generating solution (2.33), (2.34).

Here all generalized velocities and position coordinates (but not the quasicyclic ones) are periodic.

3. Systems with kinematic excitation of oscillation (the minimax criterion of stability). The minimax criterion of stability /36/ can be formulated as follows (the formulation given here incorporates certain changes, see below*). (*Strizhak T.G. Minimax criterion of stability. Preprint No.254, Inst. Electrodinamiki, Akad. Nauk SSSR, Kiev, 1981.)

Let

$$T = \frac{1}{2} \sum_{m=1}^{n} \sum_{j=1}^{n} a_{1mj}(q, u) q_m^{\cdot} q_j^{\cdot} + \frac{1}{2} \sum_{m=1}^{s} \sum_{j=1}^{s} a_{2mj}(q, u) u_m^{\cdot} u_j^{\cdot} + \sum_{m=1}^{n} \sum_{j=1}^{s} a_{3mj}(q, u) q_m^{\cdot} u_j^{\cdot}, \quad \Pi = \Pi(q, u) \tag{3.1}$$

be the kinetic and potential energy of the system described by $n + s$ generalized coordinates $q_1, \ldots, q_n, u_1, \ldots, u_s$, where the coordinates $u_1, \ldots, u_s$ are specified in the form of finite sums (μ is a small parameter)

$$u_j = \mu \sum_k u_{j,k}(q_1, \ldots, q_n, \mu) \exp(i\nu_k \omega t) \quad (j = 1, \ldots, s)$$
$$\nu_k \neq 0, \nu_{-k} = -\nu_k, \quad u_{j,-k} = \bar{u}_{j,k}, \quad \omega = 1/\mu \tag{3.2}$$

We assume that the viscous friction forces R_r $(r = 1, \ldots, n)$ corresponding to the variables $q_1, \ldots, q_n$, have the form

$$R_r = \sum_{j=1}^{n} \beta_{rj} q_j^{\cdot} + \sum_{j=1}^{s} \beta'_{rj} u_j^{\cdot}$$

where the matrix of the coefficients β_{rj} is positive definite.

We shall understand by the term quasi-equilibria of a system almost periodic motions of the form $q_j = q_j^\circ + \mu\psi_j(\omega t)$ $(j = 1, \ldots, n)$, where $q_j^\bullet$ are constants and $\langle\psi_j(\omega t)\rangle = 0$, i.e. motions representing small, high-frequency oscillations near the position $q_j = q_j^\circ$. We can also say that the quasi-equilibria correspond to the positions of equilibrium for the slow components of the motion (see below).

In this case the following theorem holds: if at some point $q_1 = q_1^\circ, \ldots, q_n = q_n^\circ$ the function $\langle \min_{q^\cdot} L(t, q, q^\cdot)\rangle$ where $L(t, q, q^\cdot)$ is the Lagrangian of the system constructed taking expressions (3.2) into account, has a coarse maximum, then we have for this point, for sufficiently small μ, an asymptotically stable quasi-equilibrium of the system.

The minimax criterion of stability was proved for canonical systems /36, 37/ using the asymptotic method. Here we shall obtain this criterion using Malkin's theorem, quoted in Sect.2.2.

After substituting expressions (3.2) into the expressions for the kinetic and potential energy (3.1), we obtain

$$L(t, q, q^\cdot) = T(t, q, q^\cdot) - \Pi(t, q) = \frac{1}{2} \sum_{m=1}^{n} \sum_{j=1}^{n} \Big[a_{mj}(q) + \tag{3.3}$$

$$\mu \sum_k a_{mj,k}(q) \exp(i\nu_k \omega t) \Big] q_m^\cdot q_j^\cdot + \sum_{m=1}^{n} \sum_k b_{m,k}(q) \exp(i\nu_k \omega t) q_m^\cdot - \Pi(q, 0) +$$

$$C_0(q) + \sum_{\substack{k \\ k \neq -p}} \sum_p C_{kp}(q) \exp[i(\nu_k + \nu_p)\omega t] + \mu R$$

$$a_{mj} = a_{1mj}(q, 0); \quad a_{mj,k} = \sum_{v=1}^{s} \frac{\partial a_{1mj}}{\partial u_v}\Big|_{u=0} u_{v,k}$$

$$b_{m,k} = \sum_{j=1}^{s} a_{3mj}(q, 0) u_{j,k} i\nu_k$$

$$C_0 = \frac{1}{2} \sum_{m=1}^{s} \sum_{j=1}^{s} a_{2mj}(q, 0) \sum_k u_{m,k} u_{j,-k} \nu_k^2$$

$$C_{kp} = -\frac{1}{2} \sum_{m=1}^{s} \sum_{j=1}^{s} a_{2mj}(q, 0) u_{m,k} u_{j,p} \nu_k \nu_p$$

$$R = \frac{1}{2} \sum_{m=1}^{s} \sum_{j=1}^{s} \sum_k \exp(i\nu_k \omega t) \sum_{v=1}^{s} \frac{\partial a_{2mj}}{\partial u_v}\Big|_{u=0} u_{v,k} \sum_p u_{m,p} i\nu_p \times$$

$$\exp(i\nu_p \omega t) \sum_w u_{j,w} i\nu_w \exp(i\nu_w \omega t) - \sum_{m=1}^{s} \frac{\partial \Pi}{\partial u_m}\Big|_{u=0} \times$$

$$\sum_k u_{m,k} \exp(i\nu_k \omega t) + \sum_{m=1}^{n} \sum_{j=1}^{s} \sum_k \exp(i\nu_k \omega t) \times$$

$$\sum_{v=1}^{s} \frac{\partial a_{3mj}}{\partial u_v}\Big|_{u=0} u_{v,k} \sum_p u_{j,p} i\nu_p \exp(i\nu_p \omega t) q_m^\cdot + O(\mu)$$

Expression (3.3) was written under the condition that $u_{j,k}$ is independent of the variables $q_1, \ldots, q_n$ and the parameter μ. In he case when $u_{j,k} = u_{j,k}(q_1, \ldots, q_n, \mu)$ the structure of the Lagrangian (3.3) is retained, only the expressions for $a_{mj,k}$ and R change, and, as we shall see below, this does not affect the final result.

We note that in the present paper the kinematic excitation of oscillations means the excitation of the system under which the law of oscillation of some of the generalized coordinates can be assumed to be given. In /36, 37/ the oscillations are introduced in a somewhat different manner, although the expression for Lagrange's function has a form analogous to (3.3). We should also not that there are no non-potential forces in the system discussed in /36, 37/, and the discussion concerns the formal stability of the quasi-equilibrium positions. The introduction of dissipative forces enables us to formulate the criterion of asymptotic stability of the system.

Making the change of variables

$$q_j = x_j, \; q_j' = \mu y_j \; (j = 1, \ldots, n)$$

we can write the equations of motion of the system in the form

$$x_j' = X_j(x, y, \tau) + \mu F_j(x, y, \tau, \mu) \tag{3.4}$$
$$y_j' = X_{j+n}(x, y, \tau) + \mu F_{j+n}(x, y, \tau, \mu) \quad (j = 1, \ldots, n)$$

Here

$$X_j = 0, \quad F_j = y_j \tag{3.5}$$
$$X_{j+n} = -\sum_{r=1}^{n} a_{jr}^{-1}(x) \sum_{k} b_{r,k}(x) i\nu_k \exp(i\nu_k\tau)$$
$$F_{j+n} = \sum_{r=1}^{n} a_{jr}^{-1}(x) \Big[-\sum_{m=1}^{n} \sum_{k} a_{rm,k}(x) i\nu_k \exp(i\nu_k\tau) y_m +$$
$$\sum_{m=1}^{n} \sum_{k} a_{rm,k}(x) \exp(i\nu_k\tau) \sum_{v=1}^{n} a_{mv}^{-1}(x) \sum_{p} b_{v,p}(x) \times$$
$$i\nu_p \exp(i\nu_p\tau) - \sum_{m=1}^{n} \sum_{k} \exp(i\nu_k\tau) \left(\frac{\partial b_{r,k}}{\partial x_m} - \frac{\partial b_{m,k}}{\partial x_r} \right) y_m -$$
$$\sum_{v=1}^{n} \sum_{m=1}^{n} \left(\frac{\partial a_{vr}}{\partial x_m} - \frac{1}{2} \frac{\partial a_{vm}}{\partial x_r} \right) y_v y_m - \frac{\partial \Pi}{\partial x_r} + \frac{\partial C_0}{\partial x_r} -$$
$$H_r(x, \tau) - \sum_{m=1}^{n} \beta_{rm} y_m \Big] + O(\mu) \quad (j = 1, \ldots, n)$$

H_r are known functions of rapid time $\tau = \omega t$ and the generalized coordinates $q_1, \ldots, q_n$, $\langle H_r \rangle = 0$; a_{rj}^{-1} are the elements of the matrix, inverse to the matrix of the coefficients a_{rj}, and the prime denotes differentiation with respect to τ.

The generating system corresponding to Eqs.(3.4)

$$x_j^{\circ\prime} = 0, \quad y_j^{\circ\prime} = -\sum_{r=1}^{n} a_{jr}^{-1}(x^\circ) \sum_{k} b_{r,k}(x^\circ) i\nu_k \exp(i\nu_k\tau) \tag{3.6}$$
$$(j = 1, \ldots, n)$$

has a family of almost periodic solutions depending on $2n$ arbitrary parameters $Q_1, \ldots, Q_n, \gamma_1, \ldots, \gamma_n$

$$x_j^\circ = Q_j, \quad y_j^\circ = -\sum_{r=1}^{n} a_{jr}^{-1}(Q) \sum_{k} b_{r,k}(Q) \exp(i\nu_k\tau) + \gamma_j \tag{3.7}$$
$$(j = 1, \ldots, n)$$

The following equations in variations correspond to the generating system (3.6) and the generating solution (3.7):

$$z_j' = 0, \quad z_{j+n}' = -\sum_{k} \sum_{m=1}^{n} \frac{\partial}{\partial Q_m} \left(\sum_{r=1}^{n} a_{jr}^{-1} b_{r,k} \right) i\nu_k \exp(i\nu_k\tau) z_m \tag{3.8}$$
$$(j = 1, \ldots, n)$$

The periodic solutions of the system linked with (3.8) and satisfying conditions (2.21), have the form

$$z_{j,s}^* = \delta_{js}, \quad z_{j,s+n}^* = \frac{\partial}{\partial Q_j} \left(\sum_{r=1}^{n} a_{sr}^{-1}(Q) \sum_{k} b_{r,k}(Q) \exp(i\nu_k\tau) \right) \tag{3.9}$$
$$z_{j+n,s}^* = 0, \quad z_{j+n,s+n}^* = \delta_{js} \quad (j = 1, \ldots, n; \; s = 1, \ldots, n)$$

Taking into account

$$\langle \psi_j'' \psi_m + \psi_j' \psi_m' \rangle = 0 \tag{3.10}$$
$$\langle i\nu_k \exp(i\nu_k\tau) \psi_j' + \exp(i\nu_k\tau) \psi_j'' \rangle = 0$$
$$\langle i\nu_k \exp(i\nu_k\tau) \psi_j + \exp(i\nu_k\tau) \psi_j' \rangle = 0$$

$$\left\langle \sum_{j=1}^{n} \psi_j' \sum_k \exp(i\nu_k\tau) \frac{\partial b_{j,k}}{\partial Q_r} + \frac{1}{2} \sum_{j=1}^{n} \sum_{m=1}^{n} \frac{\partial a_{mj}}{\partial Q_r} \psi_j' \psi_m' \right\rangle = -\frac{\partial}{\partial Q_r} \sum_{j=1}^{n} \sum_{m=1}^{n} a_{jm}^{-1}(Q) \sum_k b_{j,k} b_{m,-k}$$

$$\psi_r = -\sum_{j=1}^{n} a_{rj}^{-1}(Q) \sum_k \frac{b_{j,k}(Q)}{i\nu_k} \exp(i\nu_k\tau) \quad (r = 1, \ldots, n)$$

and relations (3.5) we obtain, after substituting the solution (3.9) into Eq.(2.19),

$$P_s \equiv \langle F_s \rangle = \gamma_s, \quad P_{s+n} \equiv \left\langle \sum_{j=1}^{n} F_j \frac{\partial}{\partial Q_j} \left(\sum_{r=1}^{n} a_{sr}^{-1} \sum_k b_{r,k} \right) \times \exp(i\nu_k\tau) + F_{s+n} \right\rangle = \sum_{r=1}^{n} a_{sr}^{-1} \left[-\frac{\partial D}{\partial Q_r} - \sum_{m=1}^{n} \beta_{rm}\gamma_m - \sum_{v=1}^{n} \sum_{m=1}^{n} \left(\frac{\partial a_{vr}}{\partial Q_m} - \frac{1}{2} \frac{\partial a_{vm}}{\partial Q_r} \right) \gamma_v \gamma_m \right] \quad (s = 1, \ldots, n)$$

Here

$$D = \Pi(Q, 0) - C_0(Q) + \sum_{j=1}^{n} \sum_{m=1}^{n} a_{jm}^{-1}(Q) \sum_k b_{j,k}(Q)\, b_{m,-k}(Q) \tag{3.11}$$

Therefore, the parameters of the generating solution satisfy the conditions

$$\gamma_j = 0, \; \partial D / \partial Q_j = 0 \; (j = 1, \ldots, n)$$

Let as assume that $Q_1 = Q_1^{\circ}, \ldots, Q_n = Q_n^{\circ}$ is a stationary point of the potential function D, and let us consider the conditions of stability of almost periodic solution of the system corresponding to the generating solution with parameters $\gamma_j = 0, Q_j = Q_j^{\circ}$ $(j = 1, \ldots, n)$.

The determinant on the left-hand side of Eq.(2.20) can be transformed to the form

$$\Delta(\lambda) = \left| \sum_{r=1}^{n} a_{sr}^{-1}(Q^{\circ}) \frac{\partial^2 D}{\partial Q_r \partial Q_j} \Big|_{Q=Q^{\circ}} + \lambda \sum_{r=1}^{n} a_{sr}^{-1}(Q^{\circ}) \beta_{rj} + \lambda^2 \delta_{sj} \right|$$

$$\Delta(\lambda) = 0 \tag{3.12}$$

Here is a characteristic solution which finds use in investigating the stability of the positions of equilibrium of the system

$$\sum_{j=1}^{n} a_{rj}(Q) Q_j^{\cdot\cdot} + \sum_{j=1}^{n} \sum_{m=1}^{n} \left(\frac{\partial a_{rj}}{\partial Q_m} - \frac{1}{2} \frac{\partial a_{mj}}{\partial Q_r} \right) Q_j^{\cdot} Q_m^{\cdot} = -\frac{\partial D}{\partial Q_r} - \sum_{m=1}^{n} \beta_{rm} Q_m^{\cdot} \tag{3.13}$$

$$(r = 1, \ldots, n)$$

It can be confirmed /36, 37/ that expression (3.11) for the potential function represents, apart from a quantity of the order of μ, an averaged minimum over the variables $Q_1^{\cdot}, \ldots, Q_n^{\cdot}$ of the Lagrangian (3.3) taken with the opposite sign.

If system (3.13) is free of frictional forces, then, using Lagrange's theorem of the stability of the positions of equilibrium of conservative systems we can conclude that the points of coarse minimum of the function D have corresponding stable positions of equilibrium. After incorporating the dissipative forces with full dissipation, the positions of equilibrium become asymptotically stable /28/ and all roots of Eq.(3.12) will have negative real parts. Thus the conditions of coarse minimum at the point $Q_1 = Q_1^{\circ}, \ldots, Q_n = Q_n^{\circ}$ of the potential function $D = -\langle \min_{Q^{\cdot}} L(t, Q, Q^{\cdot}) \rangle$ represent the sufficient conditions for the existence of an asymptotically stable, almost periodic solution of the initial system becoming, at $\mu = 0$, the generating solution (3.7) where $\gamma_j = 0, Q_j = Q_j^{\circ}$ $(j = 1, \ldots, n)$, and this completes the proof.

We note that if the coordinates $u_1, \ldots, u_s$ are periodic in t with period $T = O(\mu)$ and have continuous second-order derivatives in t, then the conditions following from the minimax criterion are also necessary in the sense that when the function D has no minimum at the point $Q_1 = Q_1^{\circ}, \ldots, Q_n = Q_n^{\circ}$, and under the condition that Eq.(3.12) has no zero roots, the corresponding solution of the initial system is unstable.

We shall also note that the minimax criterion of stability can also be obtained using the method of direct separation of motions /10, 11/ which enables us to represent the equation for the slow components $Q_1, \ldots, Q_n$ in the form (3.13). Here the rapid components have the form $\mu\psi_j\ (j = 1, \ldots, n)$ where ψ_j are found from relation (3.10).

4. Systems with dynamic excitation of oscillations. We shall assume that the motion of the system is described by the equations

$$\sum_{j=1}^{n} a_{rj}(q)\, q_j'' + \sum_{j=1}^{n}\sum_{m=1}^{n}\left(\frac{\partial a_{rj}}{\partial q_m} - \frac{1}{2}\frac{\partial a_{mj}}{\partial q_r}\right) q_j' q_m' + \tag{4.1}$$

$$\sum_{j=1}^{n} \beta_{rj} q_j' = -\frac{\partial \Pi}{\partial q_r} + \frac{1}{\mu}\sum_{k} f_{r,k}(q, \mu)\exp(i\nu_k \omega t) \quad (r = 1, \ldots, n)$$

$$\nu_k \neq 0, \quad \nu_{-k} = -\nu_k, \quad f_{r,-k} = \bar{f}_{r,k}, \quad \omega = 1/\mu$$

where $\Pi(q, \mu)$ is the potential energy of the system, μ is a small parameter, the matrix of inertial coefficients $a_{rj}(q)$ is assumed to be positive definite, and the last sum on the left-hand side of Eq.(4.1) represents viscous frictional forces with full dissipation.

Let us assume that the following relations hold:

$$\partial f_{m,k}/\partial q_r\,|_{\mu=0} = \partial f_{r,k}/\partial q_m\,|_{\mu=0} \tag{4.2}$$

$$(m = 1, \ldots, n; \ r = 1, \ldots, n)$$

The following theorem holds under the conditions formulated above: if at some point $q_1 = q_1^\circ, \ldots, q_n = q_n^\circ$ the function

$$D = \Pi\,|_{\mu=0} + \Pi_W \tag{4.3}$$

where

$$\Pi_W = \frac{1}{2}\sum_{j=1}^{n}\sum_{m=1}^{n} a_{jm}^{-1}(q)\sum_{k}\frac{f_{j,k}(q,0)\, f_{m,-k}(q,0)}{\nu_k^2} \tag{4.4}$$

has a coarse minimum, then an asymptotically stable quasi-equilibrium of system (4.1) will correspond to this point for sufficiently small values of μ.

Here a_{jm}^{-1} are the elements of the matrix that is inverse to the matrix of the coefficients a_{jm}.

Let us now prove the extremal criterion of stability using Malkin's theorem quoted in Sect.2.2.

Let us put $q_j = x_j,\ q_j' = \mu y_j$ (a prime denotes differentiation with respect to $\tau = \omega t$), and transform Eq.(4.1) to the form (3.4). We obtain

$$X_j = 0, \quad F_j = y_j$$

$$X_{j+n} = \sum_{r=1}^{n} a_{jr}^{-1}(x)\sum_{k} f_{r,k}(x, 0)\exp(i\nu_k\tau)$$

$$F_{j+n} = \sum_{r=1}^{n} a_{jr}^{-1}(x)\Big[-\frac{\partial\Pi}{\partial x_r}\Big|_{\mu=0} - \sum_{m=1}^{n}\beta_{rm}y_m -$$

$$\sum_{p=1}^{n}\sum_{m=1}^{n}\left(\frac{\partial a_{pr}}{\partial x_m} - \frac{1}{2}\frac{\partial a_{pm}}{\partial x_r}\right)y_p y_m + \sum_{k}\frac{\partial f_{r,k}}{\partial \mu}\Big|_{\mu=0}\exp(i\nu_k\tau)\Big] + O(\mu)$$

$$(j = 1, \ldots, n)$$

Then we can write Eqs.(2.19) for determining the parameters $Q_1, \ldots, Q_n, \gamma_1, \ldots, \gamma_n$ of the generating solution of the system as follows:

$$P_s \equiv \gamma_s = 0, \quad P_{s+n} \equiv \sum_{r=1}^{n} a_{sr}^{-1}(Q)\Big[-\frac{\partial\Pi}{\partial Q_r}\Big|_{\mu=0} - \sum_{m=1}^{n}\beta_{rm}\gamma_m - \tag{4.5}$$

$$\sum_{p=1}^{n}\sum_{m=1}^{n}\left(\frac{\partial a_{pr}}{\partial Q_m} - \frac{1}{2}\frac{\partial a_{pm}}{\partial Q_r}\right)\gamma_p\gamma_r - W_r(Q) = 0 \quad (s = 1, \ldots, n)$$

$$W_r = \frac{\partial}{\partial Q_r}\left(\frac{1}{2}\sum_{j=1}^{n}\sum_{m=1}^{n} a_{jm}^{-1}(Q)\sum_{k}\frac{f_{j,k}(Q,0)\, f_{m,-k}(Q,0)}{\nu_k^2}\right) + \tag{4.6}$$

$$\left\langle \sum_{m=1}^{n}\sum_{k}\left(\frac{\partial f_{m,k}}{\partial Q_r}\Big|_{\mu=0} - \frac{\partial f_{r,k}}{\partial Q_m}\Big|_{\mu=0}\right)\psi_m \exp(i\nu_k\tau)\right\rangle \quad (r = 1, \ldots, n)$$

$$\psi_r = -\sum_{j=1}^{n} a_{rj}^{-1}(Q) \sum_k \frac{f_{j,k}(Q,0)}{\nu_k^2} \exp(i\nu_k\tau) \tag{4.7}$$

where W_r are the so-called vibrational fores.

Let $Q_1 = Q_1^\circ, \ldots, Q_n = Q_n^\circ$ be a solution of the system

$$\partial\Pi / \partial Q_j |_{\mu=0} + W_j(Q) = 0 \quad (j = 1, \ldots, n) \tag{4.8}$$

If at the same time the algebraic equation

$$\Delta(\lambda) \equiv \left| \sum_{r=1}^{n} a_{sr}^{-1}(Q^\circ) \left(\frac{\partial^2\Pi}{\partial Q_r \partial Q_j} \bigg|_{\substack{Q=Q^\circ,\\ \mu=0}} + \frac{\partial W_r}{\partial Q_j} \bigg|_{Q=Q^\circ} \right) + \lambda \sum_{r=1}^{n} a_{sr}^{-1}(Q^\circ)\, \beta_{rj} + \lambda^2 \delta_{sj} \right| = 0 \tag{4.9}$$

has no roots with real parts equal to zero, then for sufficiently small μ the initial system (3.4) will have an almost periodic solution becoming, when $\mu = 0$, a generating solution with parameters $\gamma_j = 0, Q_j = Q_j^\circ$ $(j = 1, \ldots, n)$. If all roots of Eq.(4.9) have negative real parts, then the almost periodic solution in question of system (3.4) will be asymptotically stable.

In the general case, the oscillatory forces (4.6) may have no potential. However, if conditions (4.2) hold, then the "potential energy of the oscillatory forces" (4.6) will exist and expression (4.4) will hold for it. We shall further assume that the potential function (4.3) has a coarse minimum at the point $Q_1 = Q_1^\circ, \ldots, Q_n = Q_n^\circ$. Then $Q_1 = Q_1^\circ, \ldots, Q_n = Q_n^\circ$ will be a solution of system (4.8) and, moreover, all roots of equation (4.9) will have negative real parts. Therefore, in order for an asymptotically stable position of quasi-equilibrium of the initial system to correspond to some point $Q_1 = Q_1^\circ, \ldots, Q_n = Q_n^\circ$ under the conditions (4.2), it is sufficient that this point be the part of the coarse minimum of the potential function (4.3).

We note that the expression for a potential function analogous to (4.3) was found in /24/ in the coarse of solving the problem of the behaviour of a particle in a one-dimensional, rapidly oscillating field. However, the proposed generalization of the formula to the case of systems with many degrees of freedom holds only when the additional conditions (4.2), which ensure that a potential of vibrational forces exists, are satisfied. These conditions are essential, since they require that equations for the amplitudes of harmonic components of the field must hold. This circumstance was noted in /42/. The papers /19, 29/ were published practically simultaneously with the book /24/, and their authors obtained an expression for the potential function in the problem of the motion of a charged particle in a three-dimensional, rapidly oscillating electromagnetic field.

Vorovich /18/ discovered the existence of a potential function, i.e. of the "potential energy of the amplitudes of steady oscillations" in the course of solving the problem of the oscillations of a circular plate under the action of a random load, using asymptotic methods.

If in Eqs.(4.1) we replace the finite sums on the right-hand sides by some functions $f_r(q, \mu, t)$, periodic in t, with period $T = O(\mu)$, then, provided that Eq.(4.9) has no zero roots, the criterion in question will also yield the necessary conditions for the asymptotic stability of the corresponding quasi-equilibria of the initial system.

We note that systems in which the oscillations are excited dynamically, can also be studied using the method of direct separation of motions.

5. *Application of extremal criteria of stability.* *5.1. Reasons for the tendency of certain classes of weakly coupled dynamic objects to achieve synchronization.* Using the integral criterion of stability we have succeeded in showing the tendency to reach synchronization, i.e. the presence of at least one, stable in one or other sense, synchronous motion, for a number of important classes of dynamic objects under fairly general assumptions /10/. Such objects include objects with almost uniform rotations and almost converative objects. The scheme of proof is fairly simple: the functions $\Lambda = \langle[L]\rangle, \Lambda^{(I)} = \langle[L^{(I)}]\rangle$ and $\Lambda^{(II)} = \langle[L^{(II)}]\rangle$ appearing in expressions (2.4), (2.6), (2.7), (2.9) and (2.11) and representing the "fundamental part" of the potential function D, are periodic in $\alpha_1, \ldots, \alpha_k$ (or $\alpha_1 - \alpha_k, \ldots, \alpha_{k-1} - \alpha_k$ in the case of selfsimilar systems). Therefore, the function D will have minima under very general assumptions. In other words, we have succeeded in showing that in the space of parameters $\alpha_1, \ldots, \alpha_k$ or $\alpha_1 - \alpha_k, \ldots, \alpha_{k-1} - \alpha_k)$ "potential wells" invariably exist corresponding to stable synchronous motions. In particular, this method is used to obtain a general explanation of the more than accidental form of the appearance of the synchronisms (resonances) in the orbital motions of objects in the solar system (see also Sect.5.5.).

5.2. Application to the design of new oscillatory machines and technologies. The integral criterion of the stability of synchronous motions represents a working tool for investigating the basic schemes of a new class of vibrational machines and constructions, i.e. of machines with selfsynchronizing mechanical oscillation generators. The most important schemes were discussed in /9, 10/, and in /17/. Applications of the minimax criterion and the study of a number of systems with kinematic excitation of oscillations were discussed in /37/ and /14/.

5.3. *Generalization of the classical principle of selfbalancing.* As long ago as 1884 the Swedish engineer Laval discovered that an unbalanced disc mounted on a flexible shaft undergoes selfcentring in the supercritical range of rotation frequencies, i.e. at frequencies, i.e. at frequencies ω appreciably exceeding the frequency of free oscillations of the rotor p; its centre of gravity is situated practically on the axis of rotation, and this leads to a considerable reduction in the non-equilibrated forces transmitted to the shaft bearings. The effect is successfully utilized in machines.

The integral criterion of stability given in Sect.2.1. yields a generalization of the principle of selfbalancing to the multirotor systems, and the generalization is as follows /10/. When the mechanical oscillation generators (unbalanced rotors driven by asynchronous-type engines) have the same partial angular velocities and are positioned on a rigid body connected linearly and elastically to a fixed foundation, and provided the motion occurs at a sufficient distance from resonance, in which case the forces resisting the oscillations can be neglected, the quantity B in expression (2.4) is equal to zero and so is, in this case, the expression $\Lambda^{(II)}$ (see /9, 10/ for more detail). As a result, according to (2.7) the potential function is

$$D = D(\alpha_1, \ldots, \alpha_k) = \Lambda^{(I)} = \langle [T^{(I)}] - [\Pi^{(I)}] \rangle \tag{5.1}$$

where $[T^{(I)}]$ and $[\Pi^{(I)}]$ are the kinetic and potential energy of the body, respectively, calculated on the assumption that the rotors of the generators rotate uniformly in accordance with the rule $\varphi_s = \varphi_s^\circ = \sigma_s(\omega t + \alpha_s)$, and the body executes steady oscillations under the action of forces generated by the generators during such rotation. In the far supercritical range of frequencies, $\omega \gg \lambda_i$ (λ_i are the frequencies of free oscillations of the body) $T^{(I)} \gg \Pi^{(I)}$, and we then have

$$D = D(\alpha_1, \ldots, \alpha_k) = \langle T^{(I)} \rangle \tag{5.2}$$

Thus for the integral criterion of stability it follows that in the far supercritical range of frequencies the stable phasing of the rotors $\alpha_1^*, \ldots, \alpha_k^*$ is the phasing in which their imbalances will compensate each other (in the sense of minimizing the value of $\langle T^{(I)} \rangle$). In particular, if a phasing is possible under which $T^{(I)} = 0$, then it will be precisely this phasing that will be stable, i.e. the solid will be at rest. Such a phasing is called compensatory.

In the case of rotors with different partial velocities, when $B \neq 0$, the law formulated above is retained in the form of a certain tendency.

A more detailed investigation of expressions (2.4) and (5.1) and an analysis of the solutions of a number of problems, enable us to arrive at the following position, which can also be regarded as a generalized principle of selfbalancing of rotors.

The separate rotors, or several synchronously rotating rotors placed in a single, linear oscillating system and causing it to vibrate due to their lack of balance or by other factors reveal, in the range of rotational frequencies lying above the highest frequency of free oscillations of the system, a tendency to a weakening of the oscillations, and in the range of rotation frequencies lying below the lowest frequency of free oscillations, a tendency to increase the oscillations of the system, while the intermediate range of rotation frequencies splits into intervals in which we have an alternative tendency to selfbalancing, and to an increase in the oscillations.

Confirmation of the above, and examples of its use in producing selfbalancing devices, single foundations for a number of unbalanced machines, and other constructions, are given in /9/, where the corresponding investigations are also reviewed.

5.4. *Applications to the theory of electromechanical systems.* The integral criteria of stability formulated in Sect.2.3. can be used to study the oscillations of electromechanical systems /39, 40/. Here we shall concern ourselves with the systems of bodies including m linear conductors to which external EMF's are applied, and the resistances of the conductors are small compared with the inductive resistance, while the external EMF's are given periodic functions of time with small constant coefficients. We also assume tht the energy of the electric field can be neglected, the magnetic field can be regarded as quasistationary (here the charges q_r $(r = 1, \ldots, m)$, represent quasicycle coordinates) and the mechanical generalized coordinates $q_{m+1}, \ldots, q_n)$ as the position coordinates. Under these assumptions and the condition that the non-potential mechanical forces satisfy relations (2.35), the potential function (2.36) (where α_r are the constant components of the magnetic fluxes) will be equal to the value of magnetic field energy averaged over a period, from which the value of the mechanical kinetic potential and the energy of magnetization averaged over a period have been subtracted.

5.5. *Applications to the problem of resonances (synchronisms) in the motion of celestial objects.* The problems of resonances (synchronisms), which are amongst the basic problems of many bodies, engaged the attention of a considerable number of classical, as well as contemporary workers. Here we shall focus our attention preferably on the properties relevant to the problem and ensuing from the results given in Sect.2. As we said before, the results can also be obtained using the classical Poincaré /35/ and Lyapunov /26/ methods.

In 1973 a so-called principle of least interaction was formulated purely heuristically in /44/. A satellite or a planetary system of N bodies moving under the action of gravitational forces of attraction will remain, for most of the time, in a configuration for which the time-averaged value of the force function of the perturbations will be a minimum, and this configuration will be resonant, i.e. we shall have a clearly defined commensurability between the averaged motions (conversion frequencies). The force function of the perturbations is given by the relation

$$U_p = \sum_{i=1}^{N} \sum_{j>i}^{N} f \frac{m_i m_j}{\rho_{ij}}$$

where m_i, m_j are the masses of the satellites, ρ_{ij} are the distances between them, and f is the constant of attraction. The fact that it is precisely the resonance configuration that corresponds to the absolute minimum of interaction, was confirmed by the model computations in /44/.

The assertion of /44/ is closely related to the classical result of Poincaré /35/ from which it follows that for every stationary point (relative to the initial phases of the motions of the planets) of the averaged value of the function U_p, calculated from the unperturbed motion, there is a corresponding synchronous (resonant) average motion of the planets.

It was shown in /10, 12/ that, from the results of /32, 34/, which generalize the integral criterion of stability to embrace quasiconservative systems, it follows that the minima mentioned above of the averaged interaction potential have corresponding synchronous motions stable with respect to initial phases. In fact, in the case in question the objects appear to be mildly asynchronous and the expression for the potential function (2.11) has the form

$$D = \langle [U_p] \rangle - \langle [T^{(I)}] \rangle + B \qquad (5.3)$$

where $T^{(I)}$ is the kinetic energy of the central body and square brackets indicate that the corresponding quantities are calculated for the unperturbed (Keplerian) orbits.

If we assume that the mass of the central body is much larger than the masses $m_1, \ldots, m_N$ and the dissipative forces are small, then

$$D \approx \langle [U_p] \rangle \qquad (5.4)$$

We stress that in accordance with note 2° of Sect.2.2 we are discussing here the stability of the motions with respect to the differences in the phases of rotations of the bodies, which can be represented, in this case, by, for example, the corresponding arguments of the latitude.

For the majority of the bodies of the solar system the unperturbed orbits have small eccentricities e_s and small mutual inclinations J_{sj}. Analysis of the distribution of the function $D = \langle [U_p] \rangle$ in powers of e_s and $\nu_s = \sin^2 (J_{sj}/2)$ obtained by Veretinskii shows, that taking into account the free term of this expansion only, corresponding to circular orbits lying in a single plane, leads to the interaction of the bodies which ensure the stability only of resonances of the type 1:1, i.e. of circulations with the same average motions; taking into account the linear terms leads to resonances of the type $l:(l \pm 1)$ and $l:(l \pm 2)$, and the use of the quadratic terms leads also to resonances of the type $l:(l \pm 3)$ and $l:(l \pm 4)$; naturally, subsequent interactions are, generally speaking, weaker than the preceding ones. It follows therefore that we can offer the following classification of the orbital resonances according to their "relative strength" (for fixed l): we shall assign to zero order resonances of the type $l:l$, to the first order those of the type $l:(l \pm 1)$ and $l:(l \pm 2)$, to the second order those of the type $l:(l \pm 3)$ and $l:(l \pm 4)$, etc. (See also the classification of periodic motions according to type in Poincaré /35/).

Taking all this into account it is not surprising that the overwhelming majority of "intense" resonances in the solar system are of an order not higher than the second. We can call such resonances simple resonances. It is also natural to assume that the known hypothesis of Molchanov on the complete resonance of the orbital motions of the large planets of the solar system /10, 12/ is indeed valid for simple resonances. In this connection we note that the extremal property of the resonant motions makes possible a distinctive "check" of the hypotheses on the closeness of the motions of the planetary or satellite systems to resonant. A computer can be used to calculate the value of the function D for the motion in question, and compare it with the corresponding minimal values of this function. It is also of interest to determine the direction of the evolutionary change of the function D.

As we have already said in Sect.5.1, the results given yield a general theoretical explanation for the fact of the relatively frequent encounter with resonances in the orbital motions of celestial bodies, i.e. with the tendency to establish synchronism. One of the achievements of this concept is the hypothesis on the resonant character of the rings of Uranus, according to which the positions of these rings are determined by the resonances of the type 1 : 2, 2 : 3 and 3 : 4, with so-far undiscovered satellites. Such satellites were indeed discovered later by Voyager-2 /20/.

So far, we have been discussing only resonances in the motions of celestial objects, since the integral criterion of stability based on formula (2.11) cannot be used in problems dealing with orbital-rotational resonances, since the character of anisochronism in the motion of a body in an orbit and in rotational motions of the body about its centre of mass is different (for the first motion we have $\sigma = -1$, and for the second motion $\sigma = +1$). At the same time the character of the anisochronism does not figure in the criterion of stability proposed in /1, 3/ (see Sect.2.2). The results of a numerical experiment given in /1/ show good agreement with the criterion.

Different, non-classical representations, were used as the basis for studying the problem of stability, resonant character and extremal properties in /41/.

5.6. *The problem of optimizing the bipedal walk.* Here it was shown that synchronous (resonant) modes have corresponding minima of the functional characterizing the energy losses in the system*. (*Beletskii V.V. and Golubitskaya M.D. Stabilization and resonance phenomena in the model problem of bipedal walk. Preprint 14, Moscow, In-t prikladnoi matematiki im. M.V. Keldysha, 1987.) The use of this approach gave a number of essential periodic modes of a walk.

The authors thank V.V. Rumyantsev and his colleagues for useful discussions and suggestions, and V.V. Kozlov for bringing to our attention the work of Poincaré presented in Sect.1 and for making a number of other suggestions.

REFERENCES

1. BELETSKII V.V. and SHLYAKHTIN A.N., Extremal properties of resonant motions. Dokl. Akad. Nauk SSSR, 231, 4, 1976.
2. BELETSKII V.V. and KASATKIN G.V., On extremal properties of resonant motions. Dokl. Akad. Nauk SSSR, 251, 1, 1980.
3. BELETSKII V.V., Extremal Properties of Resonant Motions. Stability of Motion. Analytical Mechanics. Motion Control. Nauka, Moscow, 1981.
4. BELESKII V.V., Resonance phenomena in rotational motions of artificial and natural celestial bodies, in: Dynamics of Space Apparatus and Space Research. Mashinostroyeniye, Moscow, 1986.
5. BLEKHMAN I.I., On the problem of the stability of periodic solutions of quasilinear non-selfsimilar systems with many degrees of freedom. Dokl. Akad. Nauk SSSR, 104, 6, 1948.
6. BLEKHMAN I.I., On the stability of periodic solutions of quasilinear selfsimilar systems with many degrees of freedom. Dokl. Akad. Nauk SSSR, 112, 2, 1957.
7. BLEKHMAN I.I. and LAVROV B.P., On an integral criterion of stability. PMM, 24, 5, 1960.
8. BLEKHMAN I.I., Basis for the integral criterion of stability of motion in problems of self-locking of oscillators. PMM, 24, 6, 1960.
9. BLEKHMAN I.I., Synchronization of Dynamic Systems. Nauka, Moscow, 1971.
10. BLEKHMAN I.I., Synchronization in Nature and Technology. Nauka, Moscow, 1981.
11. BLEKHMAN I.I., Development of a concept of direct separation of motions in non-linear mechanics. Modern Problems of Theoretical and Applied Mechanics. Naukova Dumka, Kiev, 1978.
12. BLEKHMAN I.I., Stability of orbital systems. Proceedings of the Third All-Union Chetayev Conference on the Stability of Motion, Analytical Mechanics and Control of Motion. Izd-vo SO Akad. Nauk SSSR, Irkutsk, 1977.
13. BLEKHMAN I.I., Generalization of the Lagrange-Dirichlet theorem on stability of positions of equilibrium to certain classes of periodic and rotational motions Procedings of the 9-th International Conference on Non-linear Oscillations, Kiev, 3, 1984.
14. BLEKHMAN I.I. and MALAKHOVA O.Z., On the quasi-equilibrium positions of a Chetayev pendulum. Dokl. Akad. Nauk SSSR, 287, 2, 1986.
15. BOGOLYUBOV N.N., On certain Statistical Methods in Mathematical Physics. Izd-vo Akad. Nauk SSSR, Kiev, 1945.
16. VALEYEV K.G. and GANIYEV R.F., Study of the oscillations of Non-linear systems. PMM, 33, 3, 1969.
17. Vibrations in Technology. A Manual, 4, Vibrational Processes and Machines. Moscow, 1981.
18. VOROVICH I.I., Certain problems of the use of statistical methods in the theory of stability of plates and shells. Proceedings of the Fourth All-Union Conference on the Theory of Shells and Plates, Izd. Akad. Nauk ArmSSR, Yerevan, 1964.
19. GAPONOV A.V. and MILLER M.A., On the application of moving potential high-frequency wells for accelerating charged particles. Zh. Eksp. Teor. Fiz., 34, 3, 1958.

20. GOR'KAVYI N.N. and FRIDMAN A.M., On the discovery by Voyager-2 of predicted satellites governing the resonant character of Uranus rings. Pis'ma v Astron. Zh. 13, 3, 1987.
21. GURTOVNIK A.S. and NEIMARK YU.I., On the synchronization of dynamic systems. PMM, 38, 5, 1974.
22. KOZLOV V.V., Averaging in the neighbourhood of stable periodic motions. Dokl. Akad. Nauk SSSR, 264, 3, 1982.
23. LAVROV B.P., Novel formulation of the integral criterion of stability of synchronous motions of mechanical oscillators and its applications. Proceedings of the Conference on Vibrational Techniques, NIIinfstroidorkommunmash, Moscow, 1966.
24. LANDAU L.D. and LIFSHITZ E.M., Theoretical Physics, 1, Mechanics. Fizmatgiz, Moscow, 1958.
25. LUR'YE A.I., Certain problem of selfsynchronization. Proceedings of the Fifth International Conference on Non-linear Oscillations, Inst. Mat. AKad. Nauk, UkrSSR, Kiev, 3, 1970.
26. LYAPUNOV A.M., General Problem of Stability of Motion. Gostekhizdat, Moscow-Leningrad, 1950.
27. MALKIN I.G., Certain Prolems of the Theory of Non-linear Oscillations. Gostekhizdat, Moscow, 1956.
28. MERKIN D.R., Introduction to the Theory of Stability of Motion. Nauka, Moscow, 1976.
29. MILLER M.A., Motion of charged particles in high-frequency electromagnetic fields. Izv. Vuz. Radiofizika, 1, 3, 1958.
30. MITROPOL'SKII YU.A., The Method of Averging in Non-linear Mechanics. Naukova Dumka, Kiev, 1971.
31. NAGAYEV R.F., Synchronization in a system of essentially non-linear objects with a single degree of freedom. PMM, 29, 2, 1965.
32. NAGAYEV R.F., General problem of synchronization in an almost conservative system. PMM, 29, 5, 1965.
33. NAGAYEV R.F. and KHODZHAYEV K.SH., Synchronous motions in a system of objects with active constraints. PMM, 31, 4, 1967.
34. NAGAYEV R.F., The case of a generating family of quasiperiodic solutions in the theory of a small parameter. PMM, 37, 6, 1973.
35. POINCARÉ A., Collected Works. 1, New methods in celestial mechanics, Nauka, Moscow, 1971.
36. STRIZHAK T.G., The Method of Averaging in Problems of Mechanics. Donetsk, Vishcha Shk., Kiev, 1982.
37. STRIZHAK T.G., Asymptotic Method of Normalization. Vishcha Shk., Kiev, 1984.
38. KHODZHAYEV K.SH., Synchronization of mechanical oscillators connected with a linear oscillatory system. Inzh. Zh., MTT, 4, 1967.
39. KHODZHAYEV K.SH., Integral criterion of stability for systems with quasicyclic coordinates and energy relationships during the oscillations of current-carrying conductors. PMM, 33, 1, 1969.
40. KHODZHAYEV K.SH., Oscillations in non-linear electromechanical systems. Vibrations in Technology, Handbook, 2, Vibrations of Non-linear Mechanical Systems, Mashinostroyeniye, Moscow, 1979.
41. CHECHEL'NITSKII A.M., Extremality, Stability, Resonancy in Astrodynamics and Cosmonautics. Mashinostroyeniye, Moscow, 1980.
42. SHATALOV S.D., Separation of motions in electromechanical systems. Candidate. Dissertation, Leningrad, 1981.
43. SHINKIN V.N., Use of extremal properties of resonant modes in search of their stable resonant modes. Vestn. MGU, Ser. Vychisl. matematika i kibernetika, 2, 1981.
44. OVENDEN M.W., FEAGIN T. and GRAFF O., On the principal of least interaction action and the Laplacian satellites of Jupiter and Uranus, Celest. Mech. 8, 3, 1974.
45. SPERLING L., Selbstsynchronization unwuchtbehafteter Rotoren an elastischen Ketten, Intern. Tagung Verfahren und Geräte der mech. Schwingungstechnik. Magdeburg: Techn. Hochshule otto von Guericke, Teil 2, 1965.
46. SPERLING L., Beitrag zur allgemeinen Theorie der Selbstsynchronization umlaufender Unwuchtmassen in Nichtresonanzfall, Wiss. Techn. Hochschule Otto von Guericke Magdeburg. 1, 11, 1967.

Translated by L.K.